

2016

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Geometry of Cubic Polynomials

Xavier Boesken

Advisor: Dr. Otero

April 26, 2016

Abstract

Imagine a sphere floating in 3-space. By inscribing one of its great circles within an equilateral triangle, we can use the linear projection in map to the (x,y) -plane, viewed here as the complex plane $z=x+iy$, to project the vertices of the equilateral triangle onto the roots of a given cubic polynomial $p(z)$. This discovery allows us to prove Marden's Theorem: the roots of the derivative $p'(z)$ are the foci of the ellipse inscribed in and tangent to the midpoints of the triangle in the complex plane determined by the roots of the polynomial. It also sheds light on Cardano's formula for finding the roots of the cubic $p(z)$.

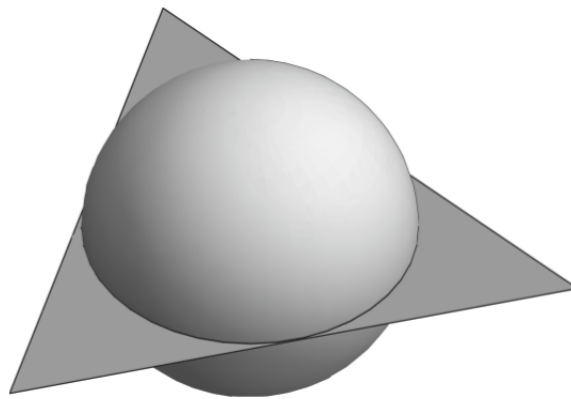


Figure 1 An equilateral triangle and its inscribed sphere

1 Introduction

In order to fully understand and prove Marden's Theorem, we need to identify several different aspects of the geometry of the floating sphere in Figure 1. What do we know about this geometric figure? Marden's Theorem states this figure projects to the roots of the polynomial, viewed as points in the complex plane. Beginning with the fundamentals, we can build a strong proof of Marden's Theorem. The fundamental theorem of algebra gives us a good starting point. Given an arbitrary polynomial:

Fundamental Theorem of Algebra. *Any polynomial of degree n has exactly n roots. The polynomial is assumed to have complex coefficients and the roots are complex as well.*

Generally, these roots are distinct, but not necessarily. For our cubic polynomial, the fundamental theorem of algebra states we have precisely three (possibly complex) roots. Let r, s, t be the roots of our cubic. Therefore, the arbitrary cubic polynomial can be written in the form

$$p(x) = a(x - r)(x - s)(x - t)$$

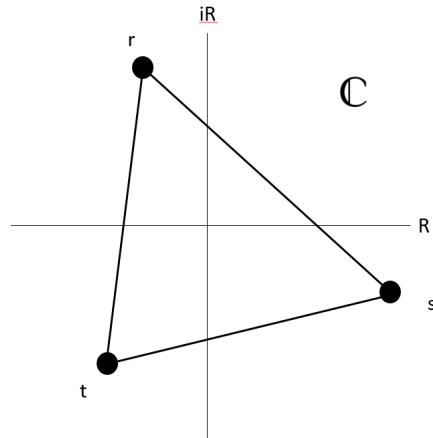
where a is a constant.

Therefore, for the arbitrary $p(z)$, the derivative of the cubic polynomial can be written as

$$p'(x) = a[(x - s)(x - t)] + a[(x - r)(x - t)] + a[(x - r)(x - s)]$$

where a is a constant.

If we were to plot these roots in the complex plane, these distinct roots would form a triangle.



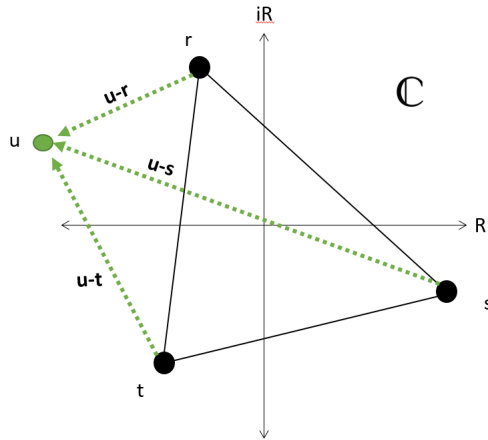
When working with such a triangle, two mathematicians came up with a theorem which says that the derivative of the cubic polynomial has roots lying within the triangle. Carl Friedrich Gauss, a German, and Felix Lucas, a Frenchman, created a theorem giving a geometric relation between a polynomial's roots and its derivative's roots.

The Gauss-Lucas Theorem. *If P is a (nonconstant) polynomial with complex coefficients, all zeros of P' belong to the convex hull of the set of zeros of P*

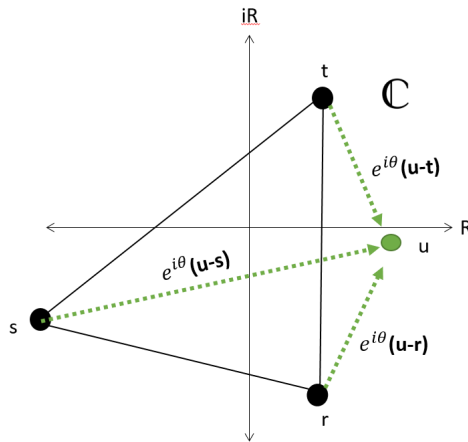
So, the Gauss-Lucas theorem implies that the two roots of the derivative of the cubic polynomial must lie within the triangle in the complex plane. We will prove the theorem for the cubic case by using proof by contradiction:

Proof. Let r, s, t be the complex roots of the cubic polynomial $p(x)$. Suppose $p'(u) = 0$ where u is **NOT** in the complex hull of r, s, t

As seen in the image below, u is not in the complex hull, meaning it is not contained within triangle formed by the roots of the polynomial. As seen below, u is divided from one of the vertices (s in our case), by one of the sides of the triangle (rt in this case).



This complex plane can be rotated by some angle such that the dividing side (rt) is vertical, with u on the right side and the third vertex (s) is on the left. We call this angle θ . Note that the vectors, drawn green in the diagram, from the roots of the polynomial to our root of the derivative are now all pointing towards the right, meaning they have positive real part.



If we look at this algebraically:

$$\begin{aligned} \frac{p'(u)}{p(u)} &= \frac{a[(u-s)(u-t)] + a[(u-r)(u-t)] + a[(u-r)(u-s)]}{a(u-r)(u-s)(u-t)} \\ &= \frac{1}{u-r} + \frac{1}{u-s} + \frac{1}{u-t} \end{aligned}$$

If we take the conjugate of both sides of the equation, we need to remember two important rules regarding conjugates:

- The conjugate of the sums is the sum of the conjugates.
- The conjugate of the reciprocal is the reciprocal of the conjugate.

So we can simplify our conjugation we get the sum of the reciprocal of the conjugate:

$$\begin{aligned}\frac{\overline{p'(u)}}{p(u)} &= \overline{\frac{1}{u-r} + \frac{1}{u-s} + \frac{1}{u-t}} \\ &= \frac{1}{\overline{u-r}} + \frac{1}{\overline{u-s}} + \frac{1}{\overline{u-t}}\end{aligned}$$

If we are to multiply each summand by 1, in the form of their respective u -root, and remember that a value multiplied by its conjugate equals the magnitude squared: e.g. $x * \bar{x} = |x|^2$:

$$\left[\frac{1}{\overline{u-r}} \cdot \frac{u-r}{u-r} \right] + \left[\frac{1}{\overline{u-s}} \cdot \frac{u-s}{u-s} \right] + \left[\frac{1}{\overline{u-t}} \cdot \frac{u-t}{u-t} \right] = \frac{u-r}{|u-r|^2} + \frac{u-s}{|u-s|^2} + \frac{u-t}{|u-t|^2}$$

If we multiply each summand by $e^{i\theta}$, we get our vectors, which we stated earlier have positive real part. The vectors are depicted in the diagram above, point to the right. Meaning, these vectors have positive real part. The contradiction comes when we recall that $p'(u) = 0$. So our result must equal zero. **BUT**, we stated that each summand has positive real part, and so must their sum. This contradicts the fact the result must be zero.

$$0 = e^{i\theta} \frac{\overline{p'(u)}}{p(u)} = e^{i\theta} \frac{u-r}{|u-r|^2} + e^{i\theta} \frac{u-s}{|u-s|^2} + e^{i\theta} \frac{u-t}{|u-t|^2}$$

This proof can be extended for polynomials of higher orders. ■

2 Marden's Projection

The sphere with the inscribed triangle is floating above a copy of the complex plane. The vertices of the triangle project onto the roots of the cubic on the (x, y) -plane, viewed as a copy of the complex numbers. The triangle, which is the image of the projection, represents the convex hull of the three roots of $p(x)$. The Gauss-Lucas theorem states the two roots of the derivative of the cubic lies within that complex hull. Knowing this, we can begin to make connections for Marden's theorem. Marden's theorem states if we have any three real numbers, not all equal, then they are the projections of the vertices of some equilateral triangle in the complex plane. For a cubic polynomial $p(x)$ with three real roots (not all equal), the inscribed circle of the equilateral triangle that projects onto those roots itself projects to an interval with endpoints equal to the roots of $p'(x)$. There is a special case where the roots are real they lie along the real axis. The equilateral triangle projects onto them must lie in the (x, z) -plane. But this special case of Marden's Theorem, which we will address later.

The theorem is saying that the roots of $p'(z)$, where z is a root of the derivative of the cubic, are the foci of the ellipse inscribed in that triangle tangent to the midpoints of the sides.

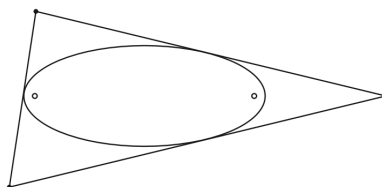
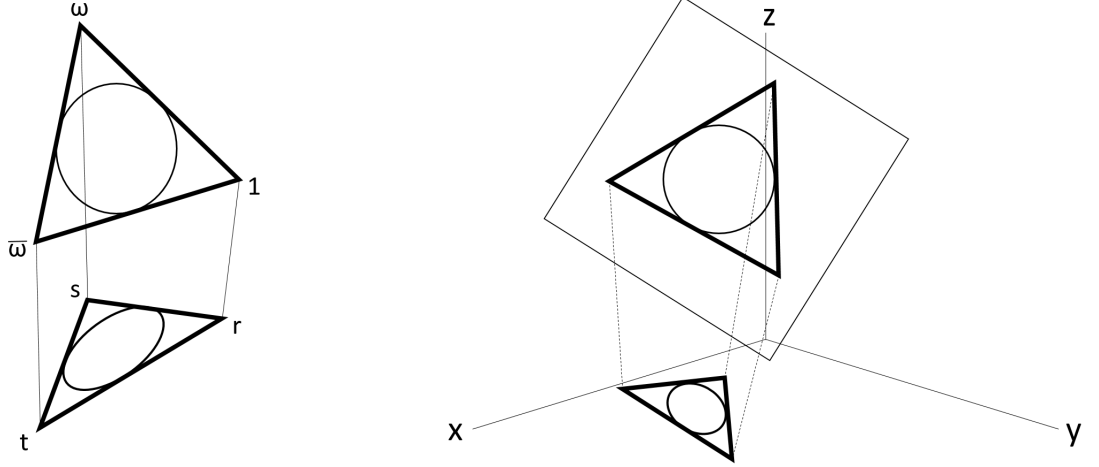


Figure 2 Roots of p' are foci of "midpoint" ellipse.

Figure 2 illustrates Marden's Theorem. As depicted in the image, the roots of p' are the foci of the midpoint of two roots of the quadratic polynomial $p(x)$.

Figure 2 shows us why it is important for our roots to be distinct and not collinear. If this were the case, our triangle would “collapse” into a line, since the roots would be collinear. This removal of the third vertex into a two vertices figure (a line) makes the ellipse within the Figure 2 collapse into a line segment contained in the same segment into which the triangle has collapsed. There would no longer be an interior space within the triangle, since the triangle would no longer exist. To better understand what this means, we need to imagine the projection of the sphere to this plane.



Consider an equilateral triangle in a copy of the complex plane. Let the vertices of the triangle be $1, \omega$, and $\bar{\omega}$. As in Figure 1, we have an inscribed sphere within the equilateral triangle. If we take the projection of the triangle at the equator of the sphere to the triangle on the complex plane, the resulting projection of that circle (that represented the equator of the sphere) is an ellipse. This illustrates a technical fact: **the projection of a circle is an ellipse**. We will need to prove that with the following lemma.

3 Projection of a Circle is an Ellipse

Recall that a linear map from \mathbb{R}^2 to itself is of the form $(x, y) \mapsto (\alpha x + \beta y, \gamma x + \delta y)$ for some α, β, γ , and $\delta \in \mathbb{R}$. If we imagine this mapping as \mathbb{C} to itself ($z := x + iy$), and if we let $a := \frac{1}{2}[(\alpha + \delta) + i(\gamma - \beta)]$ and $b := \frac{1}{2}[(\alpha - \delta) + i(\gamma + \beta)]$:

$$\begin{aligned}
 az + b\bar{z} &= \frac{1}{2}[(\alpha + \delta) + i(\gamma - \beta)](x + iy) + \frac{1}{2}[(\alpha - \delta) + i(\gamma + \beta)](x - iy) \\
 &= \frac{1}{2}[(\alpha + \delta)x + i(\gamma - \beta)iy + (\alpha + \delta)iy + (\gamma - \beta)ix + (\alpha - \delta)x - i(\gamma + \beta)iy - (\alpha - \delta)iy + (\gamma + \beta)ix] \\
 &= \frac{1}{2}[(\alpha + \delta)x - (\gamma - \beta)y + i(\alpha + \delta)y + i(\gamma - \beta)x + (\alpha - \delta)x + (\gamma + \beta)y - i(\alpha - \delta)y - i(\gamma + \beta)x] \\
 &= \frac{1}{2}[2(\alpha x + \beta y) + 2i(\gamma x + \delta y)] \\
 &= \alpha x + \beta y + i(\gamma x + \delta y)
 \end{aligned}$$

for some complex a, b

Now, we want to show that every linear map from \mathbb{C} to itself takes a unit circle to an ellipse.

Lemma. *Every one-to-one linear map $z \mapsto az + b\bar{z}$ takes the unit circle to an ellipse with foci $\pm 2\sqrt{ab}$.*

Proof. The unit circle can be parameterized by

$$C := \{e^{i\theta} : 0 \leq \theta < 2\pi\}$$

And let E be the image of the circle under the linear map $z \rightarrow az + b\bar{z}$ such that:

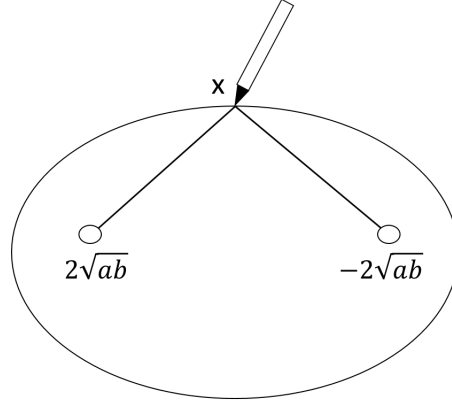
$$E := \{x : x = ae^{i\theta} + be^{-i\theta}\}$$

In other words, we are letting x be a point on this projected circle. Recall the definition of an ellipse and its foci:

An ellipse is defined by the set of all points

$$\{x \in \mathbb{C} : |x - u| + |x - v| = L\}$$

where $u, v \in \mathbb{C}$ and L is a constant



So, to prove that $\pm 2\sqrt{ab}$ are the foci of this ellipse, we shall plug them in for u, v from our definition:
Let $z := \sqrt{ae^{\frac{i\theta}{2}}}$ and $w := \sqrt{be^{-\frac{i\theta}{2}}}$

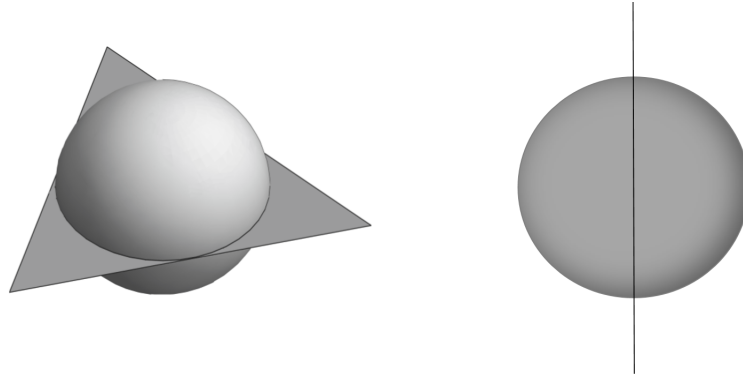
$$\begin{aligned} |x - 2\sqrt{ab}| + |x + 2\sqrt{ab}| &= |ae^{i\theta} + be^{-i\theta} - 2\sqrt{ab}| + |ae^{i\theta} + be^{-i\theta} + 2\sqrt{ab}| \\ &= |ae^{i\theta} - 2\sqrt{ab} + be^{-i\theta}| + |ae^{i\theta} + 2\sqrt{ab} + be^{-i\theta}| \\ &= |(\sqrt{ae^{\frac{i\theta}{2}}} - \sqrt{be^{-\frac{i\theta}{2}}})(\sqrt{ae^{\frac{i\theta}{2}}} - \sqrt{be^{-\frac{i\theta}{2}}})| + |(\sqrt{ae^{\frac{i\theta}{2}}} + \sqrt{be^{-\frac{i\theta}{2}}})(\sqrt{ae^{\frac{i\theta}{2}}} + \sqrt{be^{-\frac{i\theta}{2}}})| \\ &= (z - w)(z - w) + (z + w)(z + w) \\ &= |(z - w)^2| + |(z + w)^2| \\ &= (z - w)\overline{(z - w)} + (z + w)\overline{(z + w)} \\ &= (z - w)(\bar{z} - \bar{w}) + (z + w)(\bar{z} + \bar{w}) \\ &= (z\bar{z} - z\bar{w} - \bar{z}w + w\bar{w}) + (z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w}) \\ &= 2|z|^2 + 2|w|^2 \\ &= 2|a| + 2|b| \end{aligned}$$

Because the result is a constant, since a, b are both constants, then $\pm 2\sqrt{ab}$ must be the foci of the ellipse.

And thus, E is an ellipse with foci $\pm 2\sqrt{ab}$ ■

One aspect of Figure 1 is that we can rotate it within 3-space. We can do such to a point where we no longer see all three vertices of the triangle. Meaning, we only see the line formed by two of the vertices. We are rotating the figure so that the plane containing the triangle is the (x, z) -plane. The Figure below

illustrates this sufficient rotation, maintaining the image of the sphere.



The ellipse is flattened into a vertical ellipse, with the two outer vertices projecting to endpoints of an interval (line). In other words, the complex plane where this sphere exists is now vertical and directly above the real axis below in the other copy of the complex plane. If we were to stand on that real axis and look up, we would see only two of the vertices. To us, this triangle would appear as a line. We know these vertices project down to the roots of our polynomial. We can then deduce the following:

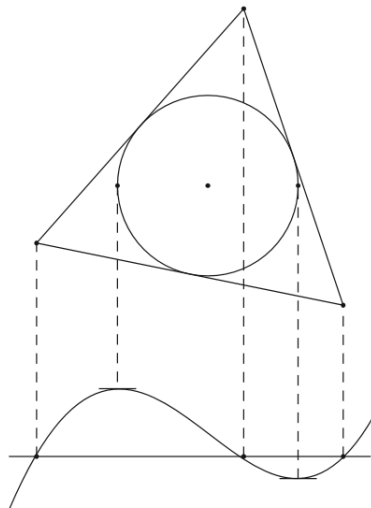


Figure 3 A cubic with corresponding triangle and circle

Theorem. Any three real numbers, not all equal, are the projections of the vertices of some equilateral triangle in the plane. For a cubic polynomial $p(x)$ with three real roots (not all equal), the inscribed circle of the equilateral triangle that projects onto those roots itself projects to an interval with endpoints equal to the roots of $p'(x)$.

Proof. Suppose we have a polynomial $p(x)$ with three real roots r, s, t . That is,

$$p(x) = (x - r)(x - s)(x - t) = x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst$$

and

$$p'(x) = 3x^2 - 2(r + s + t)x + (rs + rt + st)$$

It is verifiable that the coordinates of the vertices of the triangle are: $(r, \frac{s-t}{\sqrt{3}})$, $(s, \frac{t-r}{\sqrt{3}})$, and $(t, \frac{r-s}{\sqrt{3}})$. This is found by a form of brute force, meaning a little bit of trial and error. Using the distance formula for any

two vertices:

$$2\sqrt{\frac{r^2 + s^2 + t^2 - rs - rt - st}{3}}$$

This gives us the distance from any vertice to either of the other two. Also note that the inscribed circle has center $(\frac{r+s+t}{3}, 0)$ and the radius is $\frac{1}{\sqrt{12}}$ the distance between any two of the vertices.

$$\begin{aligned} \frac{1}{\sqrt{12}} * 2\sqrt{\frac{r^2 + s^2 + t^2 - rs - rt - st}{3}} &= \frac{1}{\sqrt{4}\sqrt{3}} * 2\sqrt{\frac{r^2 + s^2 + t^2 - rs - rt - st}{3}} \\ &= \frac{1}{2\sqrt{3}} * 2\sqrt{\frac{r^2 + s^2 + t^2 - rs - rt - st}{3}} \\ &= \frac{2}{2\sqrt{3}} * \sqrt{\frac{r^2 + s^2 + t^2 - rs - rt - st}{3}} \\ &= \frac{1}{\sqrt{3}} * \frac{1}{\sqrt{3}} * \sqrt{r^2 + s^2 + t^2 - rs - rt - st} \\ &= \frac{1}{3} * \sqrt{r^2 + s^2 + t^2 - rs - rt - st} \end{aligned}$$

Therefore, the closed interval with endpoints being the two vertices, those that project to the real axis and containing the projections of the circle, is found by adding or subtracting the value of the radius from the origin $(\frac{r+s+t}{3})$:

$$\frac{r + s + t}{3} \pm \frac{\sqrt{r^2 + s^2 + t^2 - rs - rt - st}}{3} = \frac{1}{3}[(r + s + t) * \sqrt{r^2 + s^2 + t^2 - rs - rt - st}]$$

By using the quadratic formula to find the roots of the derivative $(3x^2 - 2(r + s + t)x + (rs + rt + st))$, we get...

$$\begin{aligned} x &= \frac{2(r + s + t) \pm \sqrt{(2(r + s + t))^2 - 4(3)(rs + rt + st)}}{3(2)} \\ &= \frac{2(r + s + t) \pm \sqrt{4(r^2 + rs + rt + rs + s^2 + st + rt + st + t^2)) - 4(3)(rs + rt + st)}}{3(2)} \\ &= \frac{2(r + s + t) \pm \sqrt{4 * \sqrt{(r^2 + rs + rt + rs + s^2 + st + rt + st + t^2)} - (3rs + 3rt + 3st)}}{3(2)} \\ &= \frac{2(r + s + t) \pm 2 * \sqrt{(r^2 + rs + rt + rs + s^2 + st + rt + st + t^2) - (3rs + 3rt + 3st)}}{3(2)} \\ &= \frac{(r + s + t) \pm \sqrt{(r^2 + rs + rt + rs + s^2 + st + rt + st + t^2) - (3rs + 3rt + 3st)}}{3} \\ &= \frac{(r + s + t) \pm \sqrt{(r^2 + s^2 + t^2 - rs - rt - st)}}{3} \\ &= \frac{1}{3}[(r + s + t) * \sqrt{r^2 + s^2 + t^2 - rs - rt - st}] \end{aligned}$$

Note if we plug the values of x into $p'(x) = 3x^2 - 2(r + s + t)x + (rs + rt + st)$, we get 0. These x values are the endpoints of the interval formed by two of the vertices of the triangle formed by r, s, t that project to the real axis and contain the projections of the circle. Therefore, the endpoints of the interval are the roots of the derivative $p'(x)$. ■

Figure 3 shows the cubic polynomial and its relation to the triangle with its inscribed circle.

We can see the vertices project down to the roots of the polynomial. The outer roots create an interval on the real axis. Also seen, are the edges of the circle projecting down to the extremas. If we recall that our

extremas of our polynomials are the zeros of the derivative, then we can say the roots of the derivative are contained within the interval created by the outer roots (those visible from below).

4 Linear Maps

Given how we typically draw an ellipse (using a piece of string connecting two points), we can represent the image with the set of all points:

$$x \in \mathbb{C} : |x - u| + |x - v| = L$$

where $u, v \in \mathbb{C}$ and $L \in (|u - v|, \infty)$ is an ellipse.

This will be our definition of an ellipse, being that every ellipse is created this way. Since all ellipses have unique foci u, v and maximum length L , we can safely say such a method yields an ellipse. Now that we have defined the ellipse, we can comment on the remark of the projection of a circle is, in fact, an ellipse. In other words, the image of a circle under a linear map is an ellipse.

Recall that linear map from \mathbb{R}^2 to itself is of the form:

$$(x, y) \rightarrow (\alpha x + \beta y, \gamma x + \delta y)$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. If we were to do the same for a map from $\mathbb{C} \rightarrow \mathbb{C}$:
That is to say $z := x + iy$ maps to $\alpha x + \beta y + i(\gamma x + \delta y)$

$$\begin{aligned} \alpha x + \beta y + i(\gamma x + \delta y) &= \frac{1}{2}[(2\alpha x + 2\beta y) + i(2\gamma x + 2\delta y)] \\ &= \frac{1}{2}[(2\alpha x - 2i^2\beta y) + i(2\gamma x + 2\delta y)] \\ &= \frac{1}{2}[(\alpha x - i^2\beta y) + i(\gamma x + \delta y)] + [(\alpha x - i^2\beta y) + i(\gamma x + \delta y)] \\ &= \frac{1}{2}[(\alpha x + i\gamma x + \delta y - i^2\beta y) + (\alpha x + i\gamma x + \delta y - i^2\beta y)] \\ &= \frac{1}{2}[(\alpha + \delta + i\gamma - i\beta)x + (\alpha + \delta + i\gamma - i\beta)iy] + [(\alpha - \delta + i\gamma + i\beta)x + (\alpha - \delta + i\gamma + i\beta)(-iy)] \\ &= \frac{1}{2}[(\alpha + \delta) + i(\gamma - \beta)][x + iy] + [(\alpha - \delta) + i(\gamma + \beta)][x - iy] \\ &= \frac{1}{2}[(\alpha + \delta) + i(\gamma - \beta)][x + iy] + \frac{1}{2}[(\alpha - \delta) + i(\gamma + \beta)][x - iy] \end{aligned} \tag{1}$$

If we let:

$$\begin{aligned} a &= \frac{1}{2}[(\alpha + \delta) + i(\gamma - \beta)] \\ z &= [x + iy] \\ b &= \frac{1}{2}[(\alpha - \delta) + i(\gamma + \beta)] \\ \bar{z} &= [x - iy] \end{aligned}$$

Then we can say that every linear map is of the form $z \rightarrow az + b\bar{z}$ for some complex a, b .
This leads us into an observation of a connection between the linear map to itself and the projection of a circle.

We can also see an alternative way of looking at $p'(z)$ by the use of matrices.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix} * \underbrace{\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & t \end{pmatrix}}_{=D} = \underbrace{\begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}}_{=M} * \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix}$$

We recognize that $p(z)$ is the characteristic polynomial of the matrix D , which by the matrix equation, means that $p(z)$ is also the characteristic polynomial of the matrix M . If we call W the matrix with the omegas, then $D = W^{(-1)} \cdot M \cdot W$

$$\begin{aligned} p(z) &= \det(zI - D) \\ &= \det(zI - W^{(-1)} \cdot M \cdot W) \\ &= \det(W^{(-1)} \cdot (zI - M)W) \\ &= \det(W^{(-1)}) \cdot \det(zI - M) \cdot \det(W) \\ &= \det(zI - M) \\ &= \det\left(z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} z & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix} - \begin{pmatrix} 0 & a & b \\ b & 0 & a \\ a & b & 0 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} z-0 & 0-a & 0-b \\ 0-b & z-0 & 0-a \\ 0-a & 0-b & z-0 \end{pmatrix}\right) \\ &= \det\left(\begin{pmatrix} z & -a & -b \\ -b & z & -a \\ -a & -b & z \end{pmatrix}\right) \\ &= \begin{vmatrix} z & -a & -b \\ -b & z & -a \\ -a & -b & z \end{vmatrix} \\ &= z(z^2 - ab) - (-a)(-bz - a^2) + (-b)(b^2 + az) \\ &= z^3 - zab - zab - a^3 - b^3 - zab \\ &= z^3 - (a^3 + b^3) - 3zab \end{aligned}$$

We have convinced ourselves $p(z) = z^3 - (a^3 + b^3) - 3zab$.

5 Marden's Theorem

We are getting closer to fully understanding Marden's Theorem. However, we still need to complete the connection of the linear mapping from the equilateral figure to the projected image. Up to this point, we have only used real roots when solving for the zeros of the cubic. Now, we will suppose $p(z)$ is a cubic with distinct complex roots r, s, t . Assume that $r + s + t = 0$, for convenience.

$$p(z) = z^3 + (rs + rt + st)z - rst$$

These roots, like before, are the projections of the vertices of an equilateral triangle. Considering the Figure 1 image, there exists a linear map from the plane to itself. This mapping takes the unit circle to an ellipse.

Figure 5 illustrates how such a map would appear.

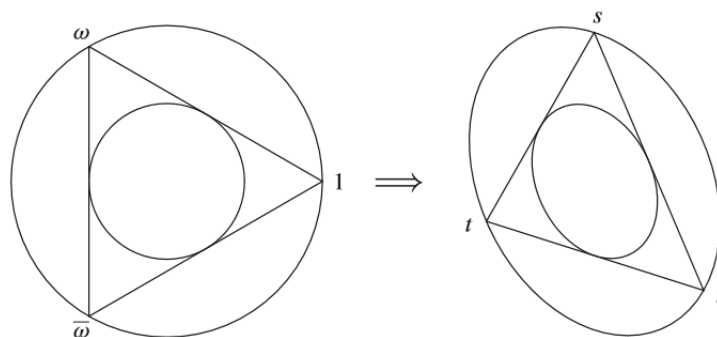
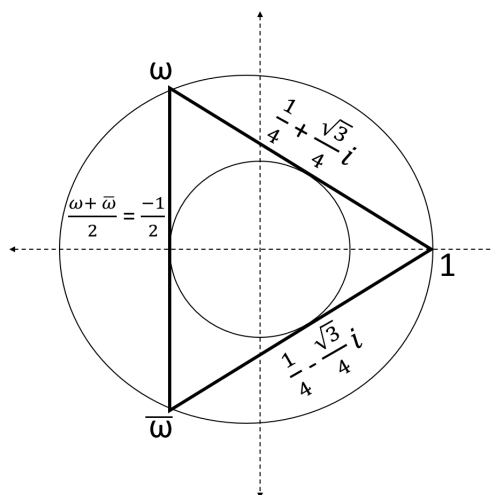


Figure 5 $f(z) = az + b\bar{z}$, where $f(1) = r$ and $f(\omega) = s$

Because the linear map of the plane to itself preserves certain characteristics of the projected image, and we know the inscribed unit circle is tangent at the midpoints of the sides of the equilateral triangle, we can say that the projection of that circle (the ellipse) is also tangent to the midpoints of the sides of the projected triangle. We can deduce that the outer radius is twice the size of the inner radius, for either image (because of the linear map). This deduction comes from the observing the geometric behavior of the figure.



The root 1 is a degree 0 and its polar form is e^0 .

The root ω is one-third of the way around the unit circle. The argument of omega is 120° or $\frac{2\pi}{3}$. We can

write ω in polar form: $\omega = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$

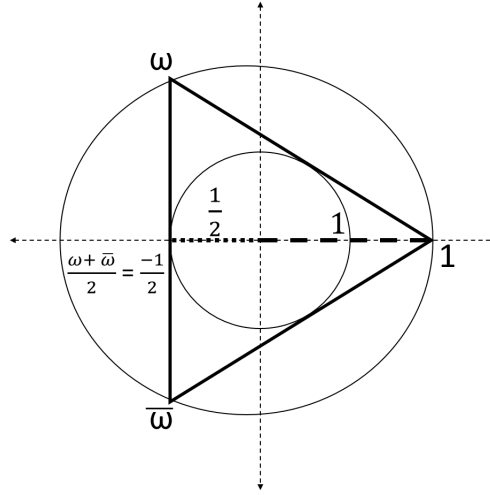
Similarly, the root $\bar{\omega}$ is two-thirds around the unit circle, or 240° or $\frac{4\pi}{3}$. We can write $\bar{\omega}$ in polar form:

$$\bar{\omega} = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$$

\therefore the midpoints of the sides of the equilateral triangle are as follows:

$$\begin{aligned} \frac{\omega + 1}{2} &= \frac{1}{4} + \frac{\sqrt{3}}{4}i \\ \frac{\omega + \bar{\omega}}{2} &= \frac{-1}{2} \\ \frac{\bar{\omega} + 1}{2} &= \frac{1}{4} - \frac{\sqrt{3}}{4}i \end{aligned}$$

Because the midpoint of $\frac{\omega + \bar{\omega}}{2} = \frac{-1}{2}$, we can see that the radius of the inner circle is $\frac{1}{2}$. This is true, because our figure is centered at the origin, and thus it is easy to see in the following figure how the inner radius is $\frac{1}{2}$ the radius of the outer radius of 1.



We can also prove this to ourselves by finding the values of a, b explicitly. We know $r + s + t = 0$ and $f(z) = az + b\bar{z}$ where $f(1) = r, f(\omega) = s$ and $f(\bar{\omega}) = t$

Because r, s, t are roots of the cubic, we can say $r + s + t = 0$. The linear mapping allows the same reasoning to follow for the projections. Meaning, $1, \omega, \bar{\omega}$ are the projections of the roots in 3-space. So, we can write them as $1 + \omega + \bar{\omega} = 0$. This can be easily verified by plugging in our evaluated values found previously.

It follows from $f(1) = r$ and $f(\omega) = s$ that $a + b = r$ and $a\omega + b\bar{\omega} = s$. We can easily verify that $1 + \omega + \bar{\omega} = 0$:

$$\begin{aligned} 1 + \omega + \bar{\omega} &= 1 + \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(\frac{-1}{2} - \frac{\sqrt{3}}{2}i\right) \\ &= 1 + \frac{-1}{2} + \frac{-1}{2} + \frac{\sqrt{3}}{2}i - \frac{\sqrt{3}}{2}i \\ &= 0 \end{aligned}$$

Since $f(1) = a(1) + b(\bar{1}) = a + b$ and $f(\omega) = a(\omega) + b(\bar{\omega})$, it follows from $r + s + t = 0$ that $t = r - s$.

$$\begin{aligned} r - s &= (a + b) - (a\omega + b\bar{\omega}) \\ &= a(1 - \omega) + b(1 - \bar{\omega}) \\ &= a(-\bar{\omega}) + b(-\omega) \\ &= a\bar{\omega} + b\omega \\ &= t \end{aligned}$$

Knowing these projected values allows us to rewrite $p(z)$ in terms of a and b .

$$\begin{aligned}
rs + rt + st &= (a + b)(a\omega + b\bar{\omega}) + (a + b)(a\bar{\omega}) + (a\omega + b\bar{\omega})(a\bar{\omega} + b\omega) \\
&= (a^2\omega + ab\bar{\omega} + ab\omega + b^2\bar{\omega}) + (a^2\bar{\omega} + ab\bar{\omega} + ab\omega + b^2\omega) + (a^2\omega\bar{\omega} + ab\omega^2 + ab\bar{\omega}^2 + b^2\omega\bar{\omega}) \\
&= a^2(\omega + \bar{\omega} + \omega\bar{\omega}) + ab(\bar{\omega} + \omega + \bar{\omega} + \omega + \omega^2 + \bar{\omega}^2) + b^2(\bar{\omega} + \omega + \omega\bar{\omega}) \\
&= a^2(\omega + \bar{\omega} + 1) + ab(3\bar{\omega} + 3\omega) + b^2(\bar{\omega} + \omega + 1) \\
&= a^2(0) + 3ab(\bar{\omega} + \omega) + b^2(0) \\
&= 3ab(-1) \\
&= -3ab
\end{aligned}$$

We can also rewrite rst in terms of a and b :

$$\begin{aligned}
rst &= (a + b)(a\omega + b\bar{\omega})(a\bar{\omega} + b\omega) \\
&= a^3\omega\bar{\omega} + a^2b\omega^2 + a^2b\bar{\omega}^2 + a^2b\omega\bar{\omega} + ab^2\bar{\omega}\omega + ab^2\omega^2 + ab^2\bar{\omega}^2 + b^3\bar{\omega}\omega \\
&= a^3(1) + a^2b(\bar{\omega} + a^2b(\omega) + a^2b(1) + ab^2(1) + ab^2(\bar{\omega}) + ab^2(\omega) + b^3(1) \\
&= a^3 + a^2b(\bar{\omega} + \omega + 1) + ab^2(1 + \omega + \bar{\omega}) + b^3 \\
&= a^3 + a^2b(0) + ab^2(0) + b^3 \\
&= a^3 + b^3
\end{aligned}$$

Knowing $r = a + b$, $s = a\omega + b\bar{\omega}$, and $t = a\bar{\omega} + b\omega$, are the roots of $p(z)$:

$$\begin{aligned}
p(z) &= z^3 + (rs + rt + st)z - rst \\
&= z^3 - 3abz - (a^3 + b^3)
\end{aligned}$$

Alternatively, if $z := p(e^{i\theta}) = ae^{i\theta} + be^{-i\theta}$

$$\begin{aligned}
z^3 - 3abz &= (ae^{i\theta} + be^{-i\theta})^3 - 3ab(ae^{i\theta} + be^{-i\theta}) \\
&= (a^3e^{i3\theta} + 3a^2be^{i\theta} + 3ab^2e^{-i\theta} + b^3e^{-i3\theta}) - 3ab(ae^{i\theta} + b^{-i\theta}) \\
&= a^3e^{i3\theta} + b^3e^{-i3\theta}
\end{aligned}$$

Then the equation will reduce to $a^3 + b^3$ when $\theta = \frac{k\pi}{3}$ with $k = 0, 2, 4$

Thus, we can say the roots of $z^3 - 3abz - (a^3 + b^3)$ are: $\{ae^{i\frac{k\pi}{3}} + be^{-i\frac{k\pi}{3}} : k = 0, 2, 4\}$

Marden's Theorem. *If $p(z)$ is a cubic polynomial with three complex roots r, s, t that form a triangle in \mathbb{C} , then the roots of $p'(z)$ are the foci of the unique ellipse tangent to the midpoints of each side.*

Proof. By the Lemma, the foci of the outer ellipse are $\pm 2\sqrt{ab}$, and so the inner ellipse has foci $\pm\sqrt{ab}$. Since $p(z) := z^3 - 3abz - (a^3 + b^3)$, and $p'(z) = 3z^2 - 3ab$. Then, $\pm\sqrt{ab}$ are the roots of $p'(z)$. ■

6 Cardano's Formula

Cardano's Formula. *The solutions of the equation*

$$z^3 - 3A - 2B = 0$$

are $a + b$, $a\omega + b\bar{\omega}$, and $a\bar{\omega} + b\omega$, where

$$a, b := \sqrt[3]{B \pm \sqrt{B^2 - A^3}}.$$

Proof. From what we did before, we know that $a + b$, $a\omega + b\bar{\omega}$, and $a\bar{\omega} + b\omega$ are roots of $z^3 - 3abz - (a^3 + b^3)$. We want to find A, B that satisfies $ab = A$ and $\frac{a^3 + b^3}{2} = B$

Provided that a, b satisfy A and B :

$$\begin{aligned} 0 &= (z - a^3)(z - b^3) \\ &= z^2 - (a^3 + b^3)z + a^3b^3 \\ &= z^2 - 2Bz + A^3 \end{aligned}$$

Using the quadratic formula for $z^2 - 2Bz + A^3$:

$$a^3, b^3 = B \pm \sqrt{B^2 - A^3}$$

And thus:

$$a, b := \sqrt[3]{B \pm \sqrt{B^2 - A^3}}.$$

■

7 Example

We can apply what we have learned from Cardano equation and Marden's Theorem to solve an example involving a cubic equation.

$$\text{Suppose } f(x) = x^3 - 3x^2 - 12x + 18.$$

$$\text{Let } p(x) = f(x + 1)$$

This will suppress the cubic and remove the x^2 from the equation. This will make our calculations easier.

$$\begin{aligned} p(x) &= f(x + 1) \\ &= (x + 1)^3 - 3(x + 1)^2 - 12(x + 1) + 18 \\ &= (x^3 + 3x^2 + 3x + 1) - 3(x^2 + 2x + 1) - (12x - 12) + 18 \\ &= x^3 + 3x^2 - 3x^2 + 3x - 6x - 12x + 1 - 3 - 12 + 18 \\ &= x^3 - 15x + 4 \end{aligned}$$

NOTE this takes the form $x^3 - 3Ax - 2B = 0$

$$\begin{aligned} &x^3 - 15x + 4 \\ \Rightarrow &x^3 - 3Ax - 2B \end{aligned}$$

$$\begin{aligned} \Rightarrow -3A &= -15 \\ \Rightarrow A &= 5 \end{aligned}$$

$$\begin{aligned} \Rightarrow -2B &= 4 \\ \Rightarrow B &= -2 \end{aligned}$$

Cardano's Formula allows us to find $a, b = \sqrt[3]{B \pm \sqrt{B^2 - A^3}}$. By cubing a, b , we can remove the cube root. Thus, we get the following:

$$\begin{aligned}
a^3, b^3 &= B \pm \sqrt{B^2 - A^3} \\
&= (-2) \pm \sqrt{(-2)^2 - (5)^3} \\
&= -2 \pm \sqrt{4 - 125} \\
&= -2 \pm \sqrt{-121} \\
&= -2 \pm i\sqrt{121} \\
&= -2 \pm 11i
\end{aligned}$$

Now that we have a^3 and b^3 , we need to take the cube root of the complex number $-2 \pm 11i$. To do this, we will use the following method to solve for the roots of the complex number:

We want to put the complex number $(\lambda \pm \mu i)$ in trigonometric form:

$$\tau \left[\cos\left(\arctan \frac{\mu}{\lambda}\right) + i \sin\left(\arctan \frac{\mu}{\lambda}\right) \right]$$

$$\text{where } \tau = \sqrt{\lambda^2 + \mu^2}$$

Steps. 1. Find $\tau = \sqrt{\lambda^2 + \mu^2}$

2. Find $\cos\left(\arctan \frac{\mu}{\lambda}\right) + i \sin\left(\arctan \frac{\mu}{\lambda}\right)$

3. We now have $\tau \left[\cos\left(\arctan \frac{\mu}{\lambda}\right) + i \sin\left(\arctan \frac{\mu}{\lambda}\right) \right]$

4. Find $\sqrt[3]{\tau} = \sqrt[3]{\sqrt{\lambda^2 + \mu^2}}$

5. Find $\cos\left(\frac{\arctan \frac{\mu}{\lambda}}{3}\right) + i \sin\left(\frac{\arctan \frac{\mu}{\lambda}}{3}\right)$

6. We know have $\sqrt[3]{\tau} \left[\cos\left(\frac{\arctan \frac{\mu}{\lambda}}{3}\right) + i \sin\left(\frac{\arctan \frac{\mu}{\lambda}}{3}\right) \right]$

7. Simplify

Now we can complete these steps using the complex number $(-2 \pm 11i)$

For $(-2 + 11i)$

For $(-2 - 11i)$

1. $\tau = \sqrt{(-2)^2 + 11^2} = \sqrt{125}$

1. $\tau = \sqrt{(-2)^2 + (-11)^2} = \sqrt{125}$

2. $\cos\left(\arctan \frac{11}{-2}\right) + i \sin\left(\arctan \frac{11}{-2}\right)$

2. $\cos\left(\arctan \frac{-11}{-2}\right) + i \sin\left(\arctan \frac{-11}{-2}\right)$

3. Now have $\sqrt{125}[\cos(-1.39) + i \sin(-1.39)]$

3. Now have $\sqrt{125}[\cos(1.39) + i \sin(1.39)]$

4. $\sqrt[3]{\sqrt{125}} = \sqrt{5}$

4. $\sqrt[3]{\sqrt{125}} = \sqrt{5}$

5. $\cos\left(\frac{-1.39}{3}\right) + i \sin\left(\frac{-1.39}{3}\right)$

5. $\cos\left(\frac{1.39}{3}\right) + i \sin\left(\frac{1.39}{3}\right)$

6. Now have $\sqrt{5}[\cos(-0.469) + i \sin(-0.469)]$

6. Now have $\sqrt{5}[\cos(0.469) + i \sin(0.469)]$

7. $a = -(2 - i)$

7. $b = -(2 + i)$

8. $a = -2 + i$

8. $b = -2 - i$

It is easy to verify that $a^3 = (-2 + i)^3 = (-2 + 11i)$ and that $b^3 = (-2 - i)^3 = (-2 - 11i)$. To do so, we use what we know from Cardano's Formula, that is $ab = A$. If we multiply $(-2 + i)(-2 - i)$, we get $4 - 2i + 2i - i^2$.

By the nature of i , we know $-i^2 = -(-1) = 1$. Therefore, $(-2 + i)(-2 - i) = 4 - 2i + 2i - i^2 = 4 + 1 = 5$, which is the value of A .

As stated before, the roots of the $p(z)$ are

$$a + b, a\omega + b\bar{\omega}, a\bar{\omega} + b\omega$$

We can solve for $a + b$, which will suffice in allowing us to solve for the other two roots, just by multiplying ω and $\bar{\omega}$. We performed similar steps earlier in this paper for the general ω and $\bar{\omega}$. **Remember** that $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\bar{\omega} = \frac{-1-i\sqrt{3}}{2}$.

$$\begin{aligned} a + b &= (-2 + i) + (-2 - i) \\ &= -2 - 2 + i - i \\ &= -4 \end{aligned}$$

$$\begin{aligned} a\omega + b\bar{\omega} &= (-2 + i)\left(\frac{-1 + i\sqrt{3}}{2}\right) + (-2 - i)\left(\frac{-1 - i\sqrt{3}}{2}\right) \\ &= \left(1 - i\sqrt{3} - \frac{1}{2}i + \frac{i^2\sqrt{3}}{2}\right) + \left(1 + i\sqrt{3} + \frac{1}{2}i + \frac{i^2\sqrt{3}}{2}\right) \\ &= 2 - \sqrt{3} \end{aligned}$$

$$\begin{aligned} a\bar{\omega} + b\omega &= (-2 + i)\left(\frac{-1 - i\sqrt{3}}{2}\right) + (-2 - i)\left(\frac{-1 + i\sqrt{3}}{2}\right) \\ &= \left(1 + i\sqrt{3} - \frac{1}{2}i - \frac{i^2\sqrt{3}}{2}\right) + \left(1 - i\sqrt{3} + \frac{1}{2}i - \frac{i^2\sqrt{3}}{2}\right) \\ &= 2 - i^2\sqrt{3} \\ &= 2 + \sqrt{3} \end{aligned}$$

Since $p(x) = f(x + 1)$, to find the roots of f , we need to subtract 1 from the roots of p :

$$\begin{aligned} \text{The roots of } p(x) \text{ are: } &[-4, 2 + \sqrt{3}, \text{ and } 2 - \sqrt{3}] \\ \therefore \text{ the roots of } f(x) \text{ are: } &[-5, 1 + \sqrt{3}, \text{ and } 1 - \sqrt{3}] \end{aligned}$$

8 Higher Dimensions

We can apply simliar logic we have used for our cubic to any regular tetrahedron. We can rotate and scale the image to match desired coordinates. It can be shown what a quartic would appear like and how the logic works similarly for higher dimension cases. Proof of these other cases are left to be revealed with future work.

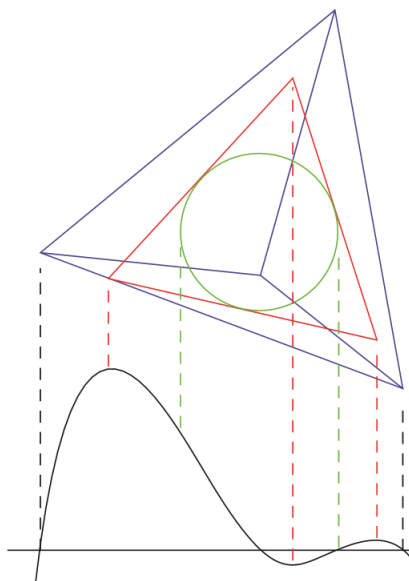


Figure 6 Tetrahedron, its inscribed sphere, and a triangle

Theorem. *Given a quartic $p(x)$ with four real roots (at least two distinct), those roots are the first coordinate projections of a regular tetrahedron in \mathbb{R}^3 . That tetrahedron has a unique inscribed sphere, which projects onto an interval whose endpoints are the two roots of $p''(x)$.*

The figure 6 above shows the quartic polynomial $p(z)$ with a regular tetrahedron, the inscribed sphere, and an equilateral triangle circumscribing that sphere. In fact, the regular tetrahedron is formed by the roots of the quartic, the equilateral triangle is formed by the roots of the derivative, and the edges of the circle project to the roots of the second derivative. We do not understand how the roots of $p(z)$, $p'(z)$, and $p''(z)$ are related geometrically, as we solve for the cubic case, but further research could allow us to solve these issues rather easily. In fact, seeing the geometric relationship of all the roots could allow us to approach open problem, such as:

Conjecture. *There does not exist a quartic polynomial p with four distinct rational roots such that p' , p'' , and p''' all have rational roots.*

This conjecture comes from the similar process we used to prove the theorem using real roots earlier. The proof for this is much more complex, and the proof good for future research and studying. Many issues like this arise from the use of higher dimensions. This would be the next step for research for this subject.

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10 Acknowledgements

I want to personally thank **Dr. Otero** for helping me with this article. His patience and understanding allowed me to complete my research. Some days were better than others, but without his help, I wouldn't have been able to write this paper. I also want to recognize **Dr. Wagner** for encouraging me to continue my Math studies. He helped me through some difficult academic times. He was a great advisor and unfortunately is leaving Xavier University. I also want to thank the **Math Department at Xavier University** for the last four years. I never had a class or teacher that was unbearable. All my professors wanted to help me learn and grow as a Mathematician. **Adam Coggeshall**, my boss, encouraged me to take charge and do the best I can. He encouraged me to take on challenges that seem daunting and tackle them head on. He inspired me to just take on any challenge I encounter, without hesitation. Lastly, I want to thank my parents, **James and Robin Boesken** for teaching me to work hard and do my best in anything I do. Their example and discipline taught me how to deal with the struggles and hardships of difficult work.

Thank you for reading my paper.